Diagnostic criterion for crystallized beams

Harel Primack* and Reinhold Blümel†
Fakultät für Physik, Albert-Ludwigs-Universität, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany
(Received 5 March 1998)

Based on a cooling hysteresis first observed in connection with small ion crystals in a Paul trap, we propose the following diagnostic criterion for establishing the presence of a crystallized beam in a storage ring: Absence of heating following reduction of the cooling power. The validity and applicability of the criterion is discussed in detail and confirmed with the help of detailed numerical simulations. [S1063-651X(98)02911-0]

PACS number(s): 29.20.Dh, 05.45.+b, 32.80.Pj, 42.50.Vk

I. INTRODUCTION

The production of crystallized beams is an important topic in storage ring physics [1,2]. As a matter of fact, crystallization has already been achieved in a miniature storage ring with essentially stationary ions [3]. Crystallization in high-energy storage rings, however, remains an elusive goal although many laboratories are working on the problem (see, e.g., [4–7]). Apart from electron cooling [8], laser cooling [9,10] is now employed by some groups [4–7] as the most efficient and most promising method to achieve the low beam temperatures required for beam crystallization. Judging from the enormous progress achieved in the past few years, success seems imminent.

But while it is straightforward to observe ion crystals in traps [11] directly with the help of cameras [12–16], it is very difficult to observe the internal structure of a fast beam in a storage ring directly with optical means. As a matter of fact, no such observation has been reported so far. Several detection schemes have been proposed, including the absence of Doppler broadening, Bragg reflection, and detection of shear waves [17]. Because of the formidable technical difficulties, none of these methods has ever been implemented. Other imaging methods may require structural changes of existing storage rings which are both technically difficult and costly. In view of these difficulties, it becomes imperative to devise diagnostic methods which are indirect and which are capable of distinguishing between a crystallized and a noncrystallized beam. In this respect, the measurement of the longitudinal temperature by fluorescence light which is emitted due to the cooling laser beam is not enough, since a longitudinally cold ion cloud can be transversally hot [4], and thus not crystallized.

It is the purpose of this paper to suggest a simple but powerful diagnostic criterion capable of making this distinction. The criterion is based on a hysteresis effect first described experimentally in connection with ion crystals in a Paul trap [13]. We propose to use this effect as a *diagnostic tool* for ion-beam crystals in storage rings. It is rooted in the observation that once an ion crystal has been produced, e.g., by laser cooling, it remains stable even as the laser cooling power is gradually reduced [16,18,19]. Since in the equilibrium reference frame the ion beam in a storage ring has similar physics to that of ions in a Paul trap, we suggest to use *absence of heating* as a diagnostic criterion for the crystalline state of an ion beam in a storage ring.

II. DIAGNOSTIC CRITERION

The above proposed diagnostic tool is based on simple observations concerning the classical dynamics and the structure of the phase space of the ion beam in a storage ring. Hence we first give an intuitive picture of the dynamical origins of the absence-of-heating criterion and also discuss its universality. Our starting point is the fact that in the absence of cooling (friction), the dynamics of the ion beam in the reference frame of the equilibrium orbit is described by a *Hamiltonian*. In general, this Hamiltonian is time dependent, due to, e.g., bending, focusing, and bunching forces, and consequently energy is not conserved. If we stroboscopically observe the phase-space point that corresponds to the beam every revolution (e.g., in a force-free straight segment of the storage ring), we get a *Poincaré section mapping* [20–22] of the motion. In general, the motion is chaotic, and the phase-space point exhibits erratic behavior, exploring large regions of phase space and on average gaining energy (rf heating) [4]. However, we expect also *fixed points* of the dynamics for which we record the same conditions every revolution of the beam. It is obvious that crystals, which are geometrically ordered structures that are static in the equilibrium orbit reference frame (according to the most restrictive definition) are of such nature, and thus correspond to fixed points of the dynamics. Moreover, if the crystal under consideration is stable under small perturbations, then we have a *stable fixed point*. According to the general theory of Hamiltonian dynamics [20–22] we know that such a stable fixed point is surrounded by a *stability (KAM) island* in which the dynamics is *confined* to ellipsoids in phase space around the fixed point [20–22] (see Fig. 1 for an illustration). This means, in turn, that the phase-space point of a beam which is close to the crystalline state remains in the vicinity of the crystalline state, and thus on average there is no gain of energy, i.e., *absence of heating*. This forms the core of the proposed diagnostic criterion: If we cool the beam enough such that it becomes close to such a stable crystal (stable fixed point) and its phase-space point resides in the stability island around the crystalline state, then the point will remain in the vicinity of the crystal even when *no cooling* is applied. Con-

*Electronic address: harel@phyc1.physik.uni-freiburg.de
†Electronic address: blumel@phyc1.physik.uni-freiburg.de

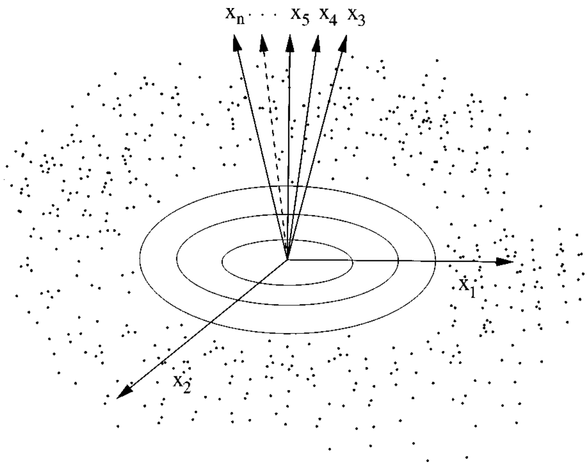

FIG. 1. Illustration of a KAM island surrounding a stable fixed point in (high-dimensional) phase space. Phase-space coordinates are denoted $x_1, x_2, \ldots, x_n$. The regular (elliptic) island is surrounded by a chaotic sea.

sequently, we shall have a crystal that persists in the absence of cooling. This is the proposed ''absence-of-heating criterion.''

In practical terms, the diagnostic criterion may be applied according to the following three steps.

(i) We start with a hot beam.
(ii) The cooling devices are switched on (e.g., electron and/or laser cooling) resulting in a cold beam whose state (crystal or not) is to be determined.
(iii) All cooling devices are switched off.

If the beam is indeed crystallized, no heating is observed and the beam remains crystallized. If the beam is just cold, but not crystallized, heating is observed after shut-down of the cooling devices.

We now discuss a few important issues that are related to the proposed diagnostic criterion and its validity. We first note that the above description of the dynamical origins of the criterion is quite general, and does not depend on the specific nature of the Hamiltonian and/or the stable fixed point. We can actually relax the above definition of a crystal and allow it to ''breathe'' as it traverses the ring, namely to change its state periodically (with the revolution frequency). Such structures will nevertheless correspond to a fixed point of the stroboscopic Poincaré mapping and thus the criterion applies to them as well. They are more realistic than the static structures considered by many authors (see, for instance, [23]) and actually were observed in numerical simulations [24]. The validity of the criterion is discussed in the following, and one may consult the logical chart depicted in Fig. 2 to follow the arguments. Concerning the necessity of the criterion, we need to show that if a crystal were actually formed when cooling is applied, then necessarily we should observe an absence of heating as the cooling is shut down. According to the above dynamical analysis this is the case, provided that the qualitative features of the dynamics are the same with and without cooling applied. In principle, this is not always the case, and the cooling forces (damping/friction) can induce structural changes to the (stability of the) dynamics. In fact, one can prove the existence of ''cooling induced freezing'' and ''cooling induced melting'' effects [25]. Indeed, the latter effect which is quite surprising was discussed in the context of Paul traps [19]. Nevertheless, since the dimensionless cooling rates (cooling rate per revolution) currently available are much smaller than unity [4], a structural change of the dynamics as the cooling is switched off is only a remote and exotic possibility that requires fine tuning of the parameters (e.g., to be close to the border of stability in the parameter space). In general, this will not happen and we thus established the necessity of the criterion. Concerning the sufficiency of the criterion, we need to show that if there is an absence of heating when no cooling is applied, then we necessarily have a crystal. Since the dynamics is chaotic, an absence of heating means that we are necessarily in the vicinity of a stable fixed point. Thus, in mathematical terms, we need to show that every stable fixed point of the dynamics corresponds to a crystal. One way to do this is simply to *define* crystals as stable fixed points of the dynamics. However, to be closer to the usual physical intuition, we need to exclude the formation of ''glassy'' (disordered) states which may also correspond to stable fixed points. In principle, one cannot exclude glasses, but if we restrict ourselves to low-density ion beams, which is the current experimental trend, we are in much better shape. Indeed, if the ion density is small enough, the beam will crystallize to a one-

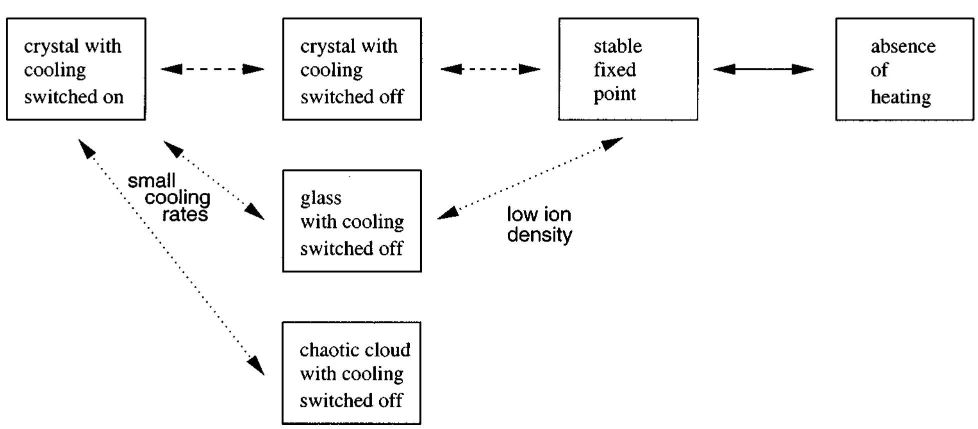

FIG. 2. Logical chart for the validity of the ''absence-of-heating'' criterion. The full double-arrowed line means ''equivalent,'' the dashed lines mean ''practically equivalent,'' and the dotted lines mean ''practically impossible.'' In the above we refer to ''glass'' as a stable fixed point of the Poincaré mapping which does not represent a geometrically ordered structure. See text for details.

dimensional linear crystal, as indicated by the work of Hasse and Schiffer [23] and by the work of Habs [17] as well as by our numerical simulations presented below. This excludes the possibility of glassy states that are expected to occur only when the ion density is much higher than the typical density for a linear crystal. Thus, if one works in a parameter range for which the linear crystal is stable, then one can safely disregard the possibility of having a cold glassy state. In summary, we have proved that for all situations of practical importance the criterion is both necessary and sufficient, thus upgrading the *observation* of an absence of heating in a Paul trap to a *criterion* for crystallization in storage rings.

In fact, the absence of heating should be even more pronounced for linear crystals in storage rings than for ions in Paul traps, since in the former case we have practically no micromotion if cooling is sufficient (since then all ions are on the symmetry axis). This cannot happen in Paul traps loaded with more than one ion since the micromotion is always present and can be quite significant for ions which are furthest from the trap's center.

Before proceeding to the numerical demonstration of the criterion, we wish to comment on the practical implementation of the criterion. The first thing to consider is that in order for the criterion to be applicable, we must have a stable fixed point for the parameters we work with. If we aim towards a linear crystal, we have to make sure that the ion density is low enough such that other stable fixed points (mainly glasses) are excluded, and also that the other parameters are tuned such that the linear crystal is stable. We are currently working on this issue, finding the appropriate regions in parameter space in which the linear crystal is stable for various focusing schemes [25]. A second point to be made is that the shut-down of the cooling should occur slowly, not abruptly. This is because we noticed in our numerical simulations that a sudden switch-off of the damping results in a phase-jump in the time dependence of the ion trajectories leading to instantaneous heating of the system that may be strong enough to disrupt the crystalline state. However, there is no problem if the shut-down is slow on the scale of the micromotion of the beam particles. A last comment concerns the beam observation time scale in a real storage ring after shut-down of the cooling devices. It is clear that even if the beam were crystallized, it would heat slowly due to, e.g., fluctuations in the confining fields and the ambient thermal radiation. When we talk about heating after shut-down of the cooling devices, we mean the fast rf heating due to the time-dependent confining fields. Since the noise processes are much slower than rf heating, there should be no problem separating the two mechanisms experimentally.

III. NUMERICAL RESULTS

We now turn to a numerical demonstration of the above proposed criterion. To do so, we modeled $N=9$ ions in a circular ring, and included the following forces: (i) Uniform magnetic dipole for constant bending; (ii) frictionlike anisotropic cooling; (iii) electrostatic harmonic confinement; (iv) cw normal quadrupole (AG) focusing; and (v) Coulomb interactions between the ions. The corresponding dimensionless equations of motion in the (local) accelerating frame that rotates with the revolution frequency are

$$\ddot{x}_i = 2\dot{z}_i - \gamma_x(\tau)\dot{x}_i + a x_i - 2q\cos(2n_f\tau)x_i + \sum_{j\neq i, j=1}^{N} \frac{(x_i - x_j)}{|\vec{r}_i - \vec{r}_j|^3}, \tag{1}$$

$$\ddot{y}_i = -\gamma_y(\tau)\dot{y}_i + a y_i + 2q\cos(2n_f\tau)y_i + \sum_{j\neq i, j=1}^{N} \frac{(y_i - y_j)}{|\vec{r}_i - \vec{r}_j|^3}, \tag{2}$$

$$\ddot{z}_i = -2\dot{x}_i - \gamma_z(\tau)\dot{z}_i - 2a z_i + \sum_{j\neq i, j=1}^{N} \frac{(z_i - z_j)}{|\vec{r}_i - \vec{r}_j|^3}. \tag{3}$$

The x direction is radial (horizontal), y is vertical, and z is the longitudinal (tangential) direction. The time and space coordinates $(t,\vec{R})$ were scaled to $(\tau,\vec{r})$ as follows:

$$\Omega t \equiv 2\tau, \quad \vec{R} \equiv l_0\vec{r}, \quad l_0 \equiv \sqrt[3]{\frac{\mathcal{Z}^2 e^2}{\pi\epsilon_0 m\Omega^2}}, \tag{4}$$

where Ω is the revolution frequency of the ions in the ring (cyclotron frequency), $\mathcal{Z}e$ is the charge of the ion, and m is its mass. The above equations are purely classical and nonrelativistic, which is justified for characteristic temperatures and ion velocities obtained, e.g., in current experiments at the Test Storage Ring (TSR) (Heidelberg) [4,5]. The parameters were chosen such as to obtain values that are typical for the TSR experiments. Therefore we used $^{+}\mathrm{Be}^{9}$ ions, $\Omega = 225$ kHz, $n_f = 8$ (dominant focusing frequency), $a = 7.1\times 10^{-3}$, and $q = 47.22$. This choice corresponds to a distance of about 75 μm$\approx 2.5 l_0$ between the ions in the 1D crystalline state and to a tune of about 2.4. We used the cooling rate as the control parameter, and considered various cases in which the cooling-power profile was

$$\gamma_z(\tau) = \gamma_0 \times \begin{cases} 1, & \tau < 100/\gamma_0 \\ \dfrac{110/\gamma_0 - \tau}{10/\gamma_0}, & 100/\gamma_0 \leqslant \tau < 110/\gamma_0 \\ 0, & 110/\gamma_0 \leqslant \tau, \end{cases} \tag{5}$$

$$\gamma_x(\tau) = \gamma_y(\tau) = \gamma_z(\tau)/5 \quad \text{(anisotropic cooling)}. \tag{6}$$

This cooling profile is composed of a very long period $(100/\gamma_0)$ of constant cooling, designed to allow for thermal equilibration of the ion beam and possible crystallization, then a gradual switch-off of the cooling, and then a time period in which there is no cooling. The latter was taken to consist of at least 10^4 revolutions $(\tau = 10^4\pi)$. We note that when no cooling is applied, the motion is Hamiltonian, and the above analysis applies. To integrate the equations of motion we used a variable-step variable-order Adams method [26].

We ran the above simulations for a wide range of the parameter γ_0. The results for $\gamma_0 = 5\times 10^{-3}$ and $\gamma_0 = 2\times 10^{-2}$ are shown in Fig. 3. In general, we observe two qualitatively different behaviors of the ion beam. When γ_0 is small $(\gamma_0 \lesssim 10^{-2})$ there is no crystallization, and the beam remains a cloud with a temperature (kinetic energy) of well-

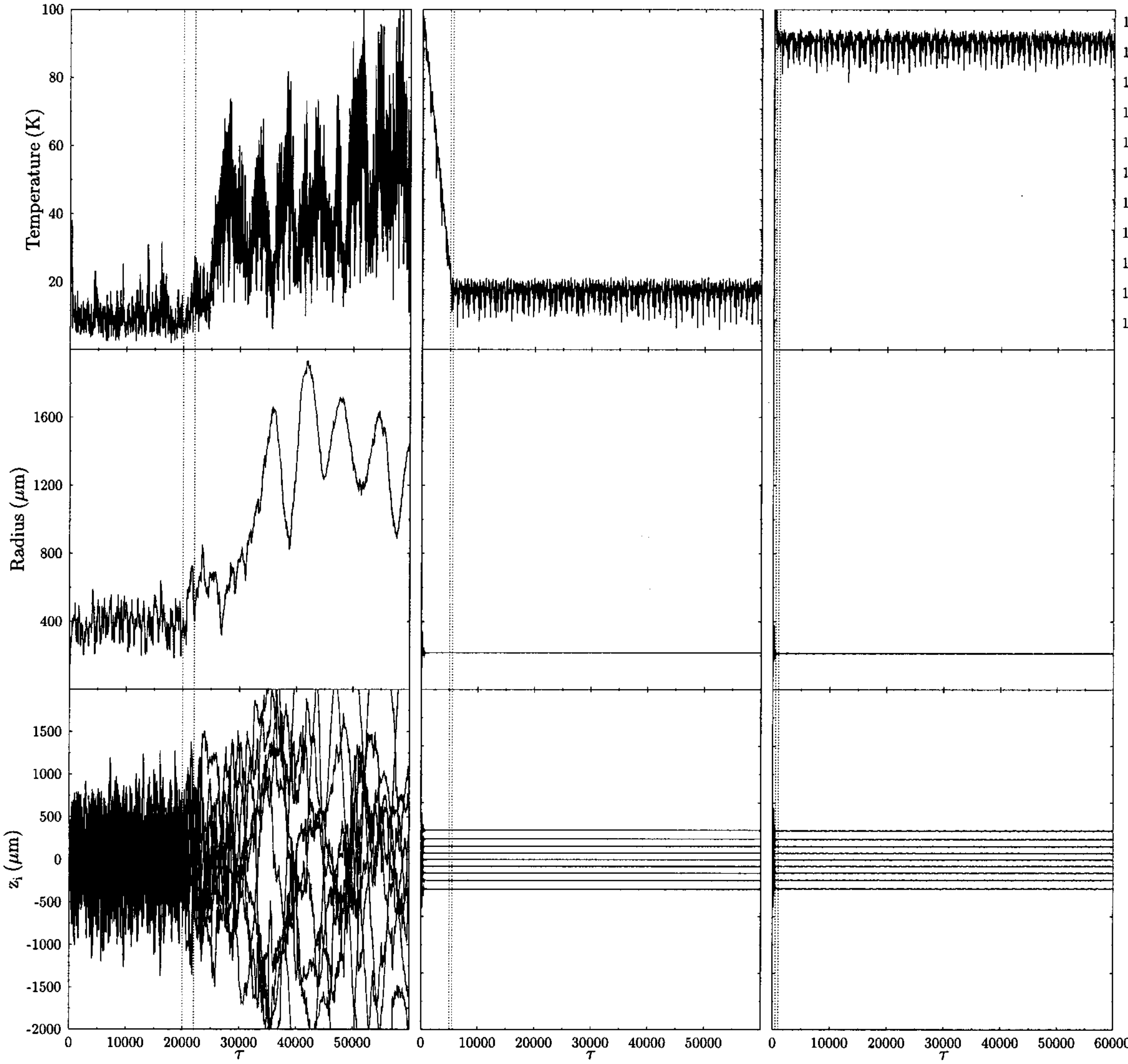

FIG. 3. Ion-beam temperatures, radii (root mean square of positions), and z positions as a function of the scaled time. Left: $\gamma_0 = 5\times10^{-3}$, cooling profile given by Eq. (6). Middle: same cooling profile with $\gamma_0 = 2\times10^{-2}$. Right: same as middle case, but with much shorter cooling period of $10/\gamma_0$. (Note the logarithmic scale for temperature in the middle and right plots.) In each case the vertical lines mark the beginning and the end of the gradual shut-down of the cooling.

defined mean as long as the cooling persists. The motion of the ions is erratic. Once cooling is shut down, an immediate heating takes place and the ion cloud expands. A representative case is shown in the left column of Fig. 3. In sharp contradistinction to the above, when γ_0 is large ($10^{-2} \lesssim \gamma_0$) we observe immediate crystallization of the beam into a well-ordered linear crystal at very low temperature. The crystalline state clearly persists when the cooling is shut down—see the middle column of Fig. 3. To further substantiate the numerical results, we ran another case, in which we took again $\gamma_0 = 2\times10^{-2}$ (strong cooling) but the cooling period consisted only of a time interval of $10/\gamma_0$. Also in this case, shown in the right column of Fig. 3, we observe crystallization that persists even when there is no cooling, although the crystal temperature is much higher than in the previous case. Bounded motion of the ions around their equilibrium positions is clearly seen. This proves that the KAM island around the crystalline state can have an appreciable size, and that an absence of heating can be achieved also for moderately low temperatures and not only for ''ideal,'' very-low-temperature situations that are beyond current cooling techniques. It should be emphasized that the above numerical results are *representative* in the sense that we considered a few (random) initial conditions for each cooling profile but always obtained the same qualitative results. This series of numerical simulations demonstrates and substantiates the proposed criterion, since it shows clearly that there is an absence of heating if and only if the beam is crystallized prior to the shut-down of the cooling.

It is interesting to note that the transition between the above scenarios of no-crystallization and crystallization is sharp as a function of γ_0. This is compatible with the findings of [24]. Also, the relatively large cooling rates needed to achieve the crystalline state are in order-of-magnitude agreement with the results of Wei, Okamoto, and Sessler [27]. In the intermediate regime ($5\times10^{-3} \lesssim \gamma_0 \lesssim 2\times10^{-2}$) we observed crystallization of the beam which did not take place immediately (time scale of order $1/\gamma_0$) but only after some time which was long compared to $1/\gamma_0$. It is important to emphasize that the exact nature of the cooling force and the cooling-power profile are immaterial as far as the proposed criterion is considered, since the criterion is based on the Hamiltonian nature of the motion that applies when there is no cooling. In other words, the cooling is used only to ''drag'' the phase-space point of the beam into various regions in phase space, and the exact way in which this dragging is performed is immaterial.

IV. DISCUSSION

Time-dependent simulations of ion-beam crystallization have been done before (see, for instance, [24,27,28]). However, in this publication we emphasize what is general about ion-beam crystallization. We do this by using the language and the methods of nonlinear dynamics. Thus, we reach a more microscopic and fundamental understanding of crystallization and heating processes. One of the central ideas is the identification of crystals with fixed points of the dynamics, and from this inferring the absence-of-heating criterion. In particular, we now understand the origin of the long survival times for cold crystals in storage rings that were reported as the results of molecular-dynamics simulations by Wei, Li, and Sessler [24] as due to the stabilizing effect of KAM islands, as discussed in Sec. II.

The absence-of-heating criterion applies to an ideal storage ring. However, conditions might be far from ideal in actual storage rings. For instance, there are unavoidable collisions with rest-gas atoms that can disrupt the crystal at low cooling rates. The situation is sketched in Fig. 4. Suppose that we start with a hot beam in a cloud state at low cooling power. For cooling rates below a critical rate, the cloud relaxes into thermodynamical equilibrium with well-defined average temperature. Increasing the cooling power we reach a point where the cloud exhibits a sharp transition into the crystalline state. Ideally, as indicated in Fig. 4 (and reflected by our simulations), the cooling power can now be reduced to zero without affecting the crystalline state (dashed line in Fig. 4). In reality, however, as already shown by the first crystallization experiment in a Paul trap [13], the crystal disrupts at some small, but finite, cooling power. This is probably due to collisions with rest-gas atoms. At this point we even arrive at a prediction for small ion crystals in Paul traps: Reducing the rest-gas pressure should extend the hysteresis towards lower cooling powers. In the limit of very small rest-gas pressure, small ion crystals may be held for long periods of time in the trap even in the absence of any cooling power. The same reasoning should apply to storage rings, if rest-gas collisions are indeed the dominant effect that disrupts crystals.

Shear forces are a central point of discussion in the context of three-dimensional beam crystals [4,17,24,28]. Because of the longitudinal motion of the beam, shear forces may lead to heating and melting of the crystal. Thus, shear forces are a major obstacle in the formation of crystallized

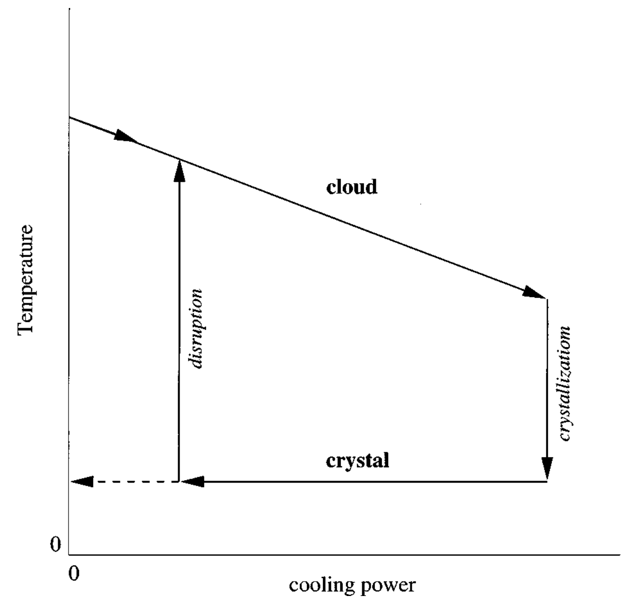

FIG. 4. Illustration of the proposed absence-of-heating criterion. It shows the phase (cloud/crystal) of the ion beam as a function of the cooling power (after the beam was thermally equilibrized). The dashed line corresponds to the ideal situation in which the crystal does not experience any sudden perturbations such as, e.g., collisions with rest-gas atoms.

beams. However, for the one-dimensional crystals treated in this publication and under active experimental investigation, shear forces are not a major concern. In any case, the effect of shear forces is included in our simulations as far as bending sections are concerned.

V. SUMMARY AND CONCLUSIONS

To summarize, in this paper we proposed a beam diagnostic criterion for deciding whether an ion beam in a storage ring is crystallized or not. The criterion is simple to apply and does not require any technical installations that are not already present in existing storage rings. With the help of model calculations, we demonstrated that the criterion works well for low-density crystallized ion chains. Since we expect that the first crystalline geometry achieved in a storage ring will be the linear chain, the absence-of-heating criterion may play an important role in proving experimentally the presence of a crystallized beam in a storage ring.

ACKNOWLEDGMENTS

The authors acknowledge fruitful discussions with R. Grimm and with M. Weidemüller. H.P. is grateful for financial support from MINERVA. R.B. is grateful for financial support by the Deutsche Forschungsgemeinschaft (Grant No. SFB 276).

[1] J. P. Schiffer and P. Kienle, Z. Phys. A **321**, 181 (1985).
[2] A. Rahman and J. P. Schiffer, Phys. Rev. Lett. **57**, 1133 (1986).
[3] G. Brikl, S. Kassner, and H. Walther, Europhys. News **23**, 143 (1992).
[4] D. Habs and R. Grimm, Annu. Rev. Nucl. Part. Sci. **45**, 391 (1995).
[5] H.-J. Miesner, R. Grimm, M. Grieser, D. Habs, D. Schwalm, B. Wanner, and A. Wolf, Phys. Rev. Lett. **77**, 623 (1996).
[6] J. S. Hangst, M. Kristensen, J. S. Nielsen, O. Poulsen, J. P.

Schiffer, and P. Shi, Phys. Rev. Lett. **67**, 1238 (1991).
[7] J. S. Hangst, J. S. Nielsen, O. Poulsen, P. Shi, and J. P. Schiffer, Phys. Rev. Lett. **74**, 4432 (1995).
[8] H. Danared, Phys. Scr. **T59**, 121 (1995).
[9] T. W. Hänsch and A. L. Schawlow, Opt. Commun. **13**, 68 (1975).
[10] S. Stenholm, Rev. Mod. Phys. **58**, 699 (1986).
[11] P. K. Ghosh, *Ion Traps* (Clarendon Press, Oxford, 1995).
[12] L. Reyna, J. Hoffnagle, R. G. DeVoe, and R. G. Brewer, Phys. Rev. Lett. **61**, 255 (1988).
[13] F. Diedrich, E. Peik, J. M. Chen, W. Quint, and H. Walther, Phys. Rev. Lett. **59**, 2931 (1987).
[14] D. J. Wineland, J. C. Bergquist, W. M. Itano, J. J. Bollinger, and C. H. Manney, Phys. Rev. Lett. **59**, 2935 (1987).
[15] R. Blümel, J. M. Chen, E. Peik, W. Quint, W. Schleich, Y. R. Shen, and H. Walther, Nature (London) **334**, 309 (1988).
[16] R. Blümel, C. Kappler, W. Quint, and H. Walther, Phys. Rev. A **40**, 808 (1989).
[17] D. Habs, in *Frontiers of Particle Beams, Lecture Notes in Physics*, edited by M. Month and S. Turner (Springer, New York, 1988).
[18] R. Blümel, J. M. Chen, F. Diedrich, E. Peik, W. Quint, and H. Walther, GSI-89-10 report, April, 1989, pp. 194–230.
[19] R. Blümel, Phys. Rev. A **51**, 620 (1996).
[20] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
[21] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, New York, 1983).
[22] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
[23] R. W. Hasse and J. P. Schiffer, Ann. Phys. (N.Y.) **203**, 419 (1990).
[24] J. Wei, X.-P. Li, and A. M. Sessler, Phys. Rev. Lett. **73**, 3089 (1994).
[25] H. Primack and R. Blümel (unpublished).
[26] *NAG Fortran Library Manual, Mark 14* (The Numerical Algorithm Group Limited, Oxford, England, 1990).
[27] J. Wei, H. Okamoto, and A. M. Sessler, Phys. Rev. Lett. **80**, 2606 (1998).
[28] J. P. Schiffer and A. Rahman, Z. Phys. A **331**, 71 (1988).